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1992 J. Phys. A: Math. Gen. 25 L1351

(http://iopscience.iop.org/0305-4470/25/24/002)

J. Phys. A: Math. Gen. 25 (1992) L1351–L1354. Printed in the UK

LETTER TO THE EDITOR

On localized solitonic solutions of a (2+1)-dimensional sine-Gordon system

W K Schief

Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, LE11 3TU, UK

Received 31 July 1992

Abstract. Exponentially decaying solutions of an integrable sine-Gordon system in 2+1 dimensions are presented. A simple superposition formula is given.

It has recently been shown that the (2+1)-dimensional integrable extension of the classical sine-Gordon equation found by Konopelchenko and Rogers [1]

$$\left(\frac{\theta_{tx}}{\sin\theta}\right)_x - \left(\frac{\theta_{ty}}{\sin\theta}\right)_y + \frac{\theta_x\tilde{\theta}_y - \theta_y\tilde{\theta}_x}{\sin^2\theta} = 0$$

$$\left(\frac{\tilde{\theta}_x}{\sin\theta}\right)_x - \left(\frac{\tilde{\theta}_y}{\sin\theta}\right)_y + \frac{\theta_x\theta_{ty} - \theta_y\theta_{tx}}{\sin^2\theta} = 0$$

is amenable to an extended version of a Darboux–Levi transformation [2]. The simplest solutions that have been obtained by this approach have the following form:

$$\theta = 4\tan^{-1}\left(\frac{|D^+|}{|D^-|}\right)$$

where

$$D^{\pm} := \begin{pmatrix} 0 & e^{-\alpha_1} & \dots & e^{-\alpha_N} \\ e^{\pm\alpha_1} & & & \\ \vdots & & M & \\ e^{\pm\alpha_N} & & & \end{pmatrix}$$

$$M_{ij} := 2\frac{\lambda_j}{\lambda_i+\lambda_j}\,e^{\alpha_i+\alpha_j} + 2\frac{\mu_j}{\mu_i+\mu_j}\cdot e^{-\alpha_i-\alpha_j}$$

and

$$\alpha_i := \tfrac{1}{2}(\lambda_i-\mu_i)x - \tfrac{1}{2}(\lambda_i+\mu_i)y + (\lambda_0\lambda_i^{-1} - \mu_0\mu_i^{-1})t + \nu_i$$

with arbitrary real constants λ_i, μ_i, λ_0, μ_0 and complex constants ν_i the imaginary part of which being half-integer multiples of π.

0305-4470/92/241351+04$07.50

In order to construct localized solutions we now restrict the parameters to $\mu_i = -\lambda_i$ for $i = 1, \ldots, N_0$ and $\mu_i = \lambda_i$ for $i = N_0 + 1, \ldots, N$, that is

$$\alpha_i = \lambda_i x + (\lambda_0 + \mu_0)\lambda_i^{-1} t + \nu_i \qquad i = 1, \ldots, N_0$$

$$\alpha_i = -\lambda_i y + (\lambda_0 - \mu_0)\lambda_i^{-1} t + \nu_i \qquad i = N_0 + 1, \ldots, N.$$

For $N = 2$, $N_0 = 1$ we get the solution

$$\theta = 4 \tan^{-1}\left(\frac{\cosh(\alpha_1 + \alpha_2)}{\sinh(\alpha_1 - \alpha_2)}\right)$$

where multiplicative factors have been absorbed into ν_1 and ν_2.

The associated quantity

$$\theta_x \theta_y = 16 \frac{\lambda_1 \lambda_2}{\cosh 2\alpha_1 \cosh 2\alpha_2} \tag{1}$$

provides a localized object since it decays to zero exponentially as $(x, y) \to \infty$ in any direction and has the amplitude $\pm 16\lambda_1\lambda_2$ depending on the imaginary parts of ν_1 and ν_2. Interestingly, it can be regarded as a nonlinear superposition of two one-dimensional (anti)kinks for positive amplitude and a kink and an antikink for negative amplitude moving in the x-direction and y-direction respectively, that is

$$\theta_x \theta_y = a_x b_y$$

where a and b are the well known (anti)kink solutions of two copies of the sine-Gordon equation, namely:

$$a_{xt} = 4(\mu_0 + \lambda_0) \sin a$$

$$b_{yt} = 4(\mu_0 - \lambda_0) \sin b.$$

In figure 1 the kink-kink solution (1) is plotted at a fixed time t.

For $N = 3$, $N_0 = 2$ it may be shown that the solution decomposes asymptotically into two solutions as described by (1) provided the quantities $(c_2 + c_{32} + c_{13}) \times (c_{21} + c_{32} + c_{13})$ and $(c_{21} + c_{32}^{-1} + c_{13}^{-1}) c_2 c_{32} c_{13}$ are negative, where $c_{ij} := (\lambda_i - \lambda_j)/(\lambda_i + \lambda_j)$. Asymptotic analysis shows that they behave like one-dimensional solitons, that is, the only indication that an interaction has occurred is that the two objects are phaseshifted. The amplitudes remain unchanged.

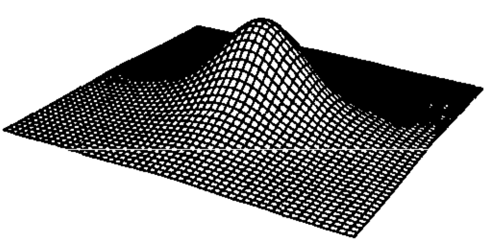

Figure 1. The kink-kink solution (1) plotted at fixed time t.

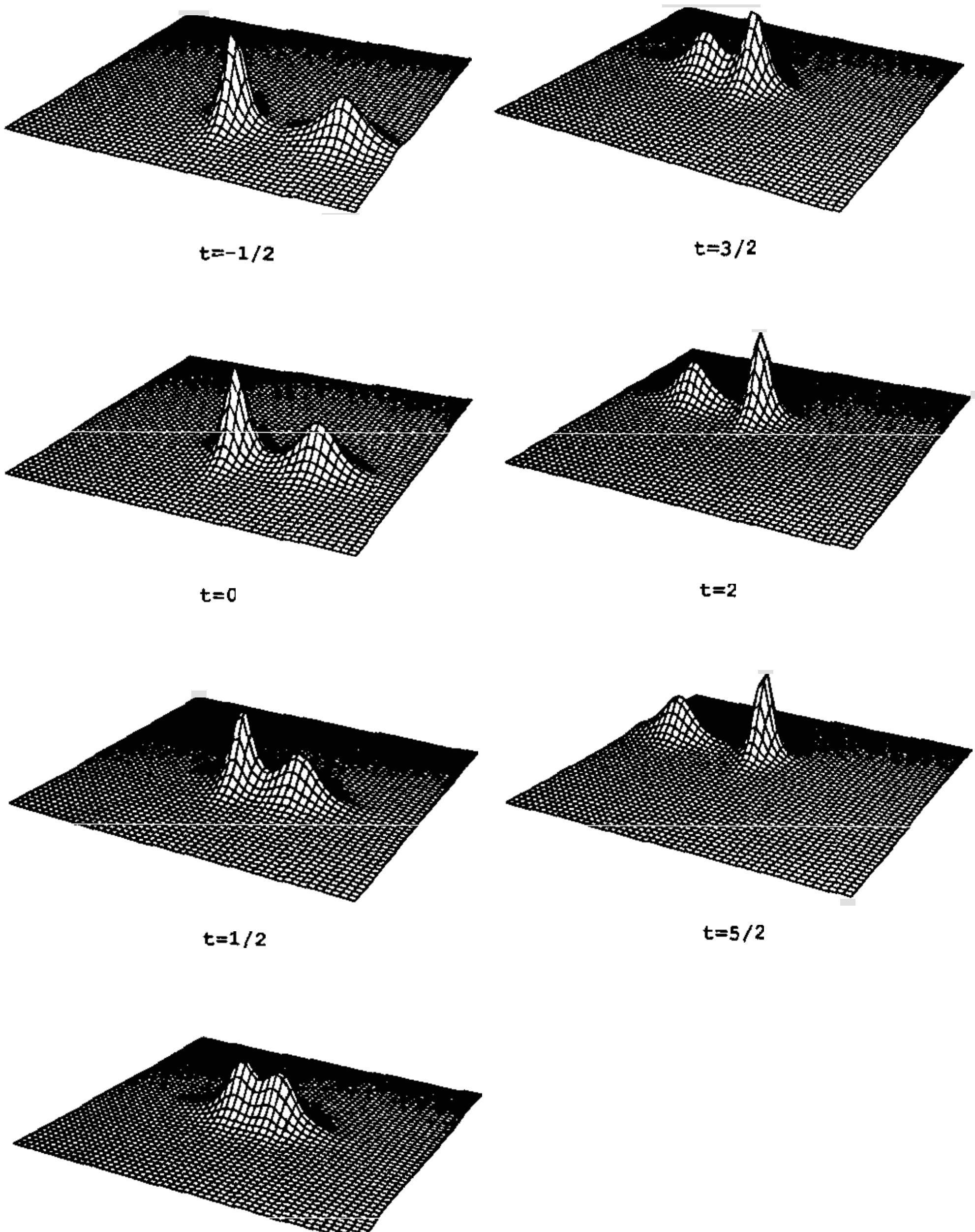

Figure 2. Interaction of a kink–kink with a kink–kink at various times.

In figures 2 and 3 snapshots are shown at various times during the interaction of a kink–kink with a kink–kink and kink–antikink respectively.

A more detailed study of these solutions is currently being undertaken.

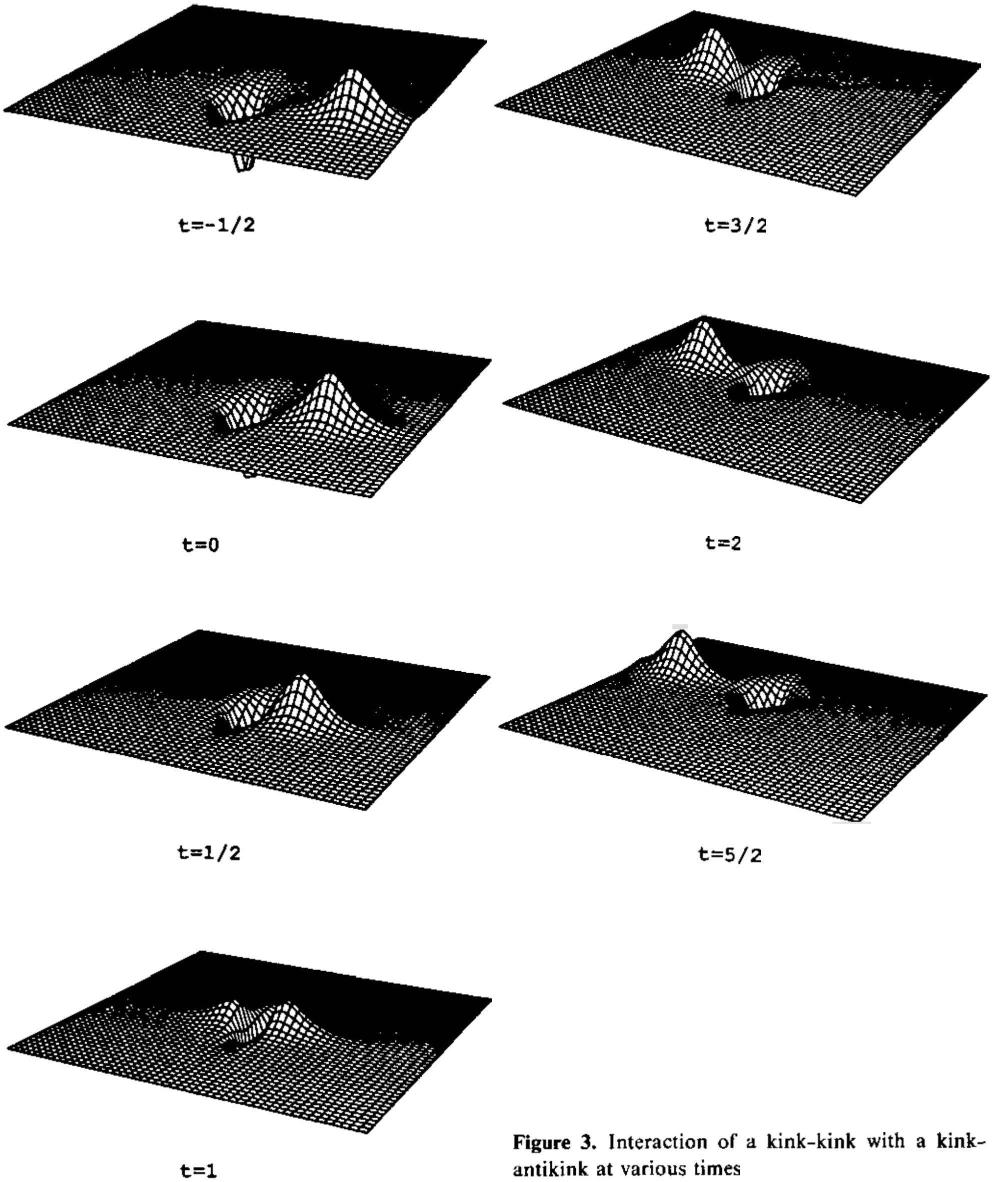

Figure 3. Interaction of a kink-kink with a kink-antikink at various times

The author would like to thank Professor C Rogers for various contributions.

References

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[2] Konopelchenko B G, Schief W K and Rogers C 1992 A (2+1)-dimensional sine-Gordon system: it's auto-Bäcklund transformation *Phys. Lett.* A in press